IRREGULAR FILTRATION IN STRATIFIED GROUNDS

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A number of theoretical investigations, carried out by P. Ya. Polubarinova-Kochina and other Soviet researchers, and the hydrodynamic evidence of the Dupuis formula, given by I. A. Charnyi, showed good agreement of the results of the hydraulic theory with the accurate hydrodynamic theory. Subsequently, many leading research workers turned their attention to the great potentialities and effectiveness of the hydraulic theory in solving practical filtration problems.

At present, owing to the works of P. Ya. Polubarinova-Kochina, I. A. Charnyi, S. F. Aver'yanov, N.N. Verigin, V. I. Aravin, S. N. Numerov, V. M. Shestakov, F. M. Bochever, and other research workers, the hydraulic theory of filtration has been widely developed and forms a firm basis of hydrogeological calculations.

In this field, P. Ya. Polubarinova-Kochina [1-3], in particular, has developed accurate solution methods and has also examined the problems of linearizing the Boussinesq equation; she has solved a number of important practical problems of filtration in interconnected strata and has investigated problems of the influence of infiltration and evaporation on the distribution of pressure heads in stratified grounds in the case of established movements.

Within the framework of the hydraulic theory, based on the hypothesis of A. N. Myatiev and N. K. Girinskii, in accordance with which the movement of liquid takes place in water-bearing strata mainly parallel to the plane of stratification (but in the slightly permeable clay strata which separates them it takes place perpendicularly to it), the filtration process in the stratified grounds is described by a closed system of differential equations of the elliptical and parabolic types [1, 2].

In deriving these equations it is assumed that in the process of interaction of the water-bearing strata the slightly permeable strata only play the part of a connecting element and, in the case of external influences (during lowering and raising of pressure heads in the water-bearing levels), the variations taking place in slightly permeable strata as a result of unimportant elastic reserves of free moisture are quite small.

For systems of water-bearing strata, interconnected by solid clay interstratifications of small thickness, obviously an assumption of this kind does not lead to large errors. Hence, the results obtained by solving the systems of equations of the hydraulic theory reflect a true picture of the filtration process in stratified grounds.

1. The system of differential equations of nonstationary filtration in interacting strata with the same slight gradient i_0 can be written in the form [1, 2]

$$\sigma_i \frac{\partial h_i}{\partial t} = k_i m_i \Delta h_i - \frac{\lambda_{i-1}}{\mu_{i-1}}(h_i - h_{i-1}) - \frac{\lambda_i}{\mu_i}(h_i - h_{i+1}) + w_i \quad (i = 1, 2, \ldots, n) \tag{1.1}$$

Here Δ is the Laplace operator, and $h_i = h_i(x, y, t)$ is the unknown pressure head relative to a certain horizontal plane,

$$w_i = \begin{cases} w(x, y, t) & \text{when} \quad i = 1 \\ 0 & \text{when} \quad i \neq 1 \end{cases}$$

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w (x, y, t) is the limiting function of infiltration, σ_1 is the effective porosity of the stratum without a pressure head, $\sigma_i = \gamma_i m_i \beta_i^*$ (i = 2, 3, ..., n), γ_i is the specific weight of the water, β_i^* is the coefficient of elasticity of the pressure strata, k_i, λ_i are the filtration coefficients, and m_i, μ_i are the capacities of the water-bearing and slightly permeable layers. In the system (1.1) the magnitude of evaporation from a free surface is approximated by the component

$$-\frac{\lambda_0}{\mu_0}(h_1 - h_0) \qquad (h_1 > h_0)$$

Here $h_0 = h_{00} + i_0 x$, and the pressure head in (n+1)-th stratum $h_{n+1} = h_{n+1,0} + i_0 x$, where h_{00}, $h_{n+1,0}$, λ_0, μ_0 are certain constants.

It must be noted that in the derivation of the first equation the systems (1.1) usually become a nonlinear equation with respect to h_1. Here it is linearized, and the water capacity of the layer $h_1 - i_0 x$ is assumed to be equal to a certain mean quantity m_1 = const.

Subsequently, changing over to the dimensionless magnitudes, we deal with the following designations:

$$h_i = \frac{h_i(x, y, t)}{m_k}, \quad \xi = \begin{cases} x / m_k \\ r / m_k \end{cases}, \quad q_i = \begin{cases} q_i(x, t) / k_i m_i \\ q_i(r, t) / 2\pi k_i m_i m_k \end{cases}$$

$$\tau = \frac{\lambda_k t}{\sigma_1 \mu_k}, \quad \alpha_i = \frac{\sigma_i}{\sigma_1}, \quad \beta_i = \frac{\lambda_i \mu_k}{\lambda_k \mu_i}, \quad a_i = \frac{k_i \mu_k}{m_i \lambda_k}, \quad \varepsilon_i = \frac{\mu_k w_i}{\lambda_k m_k} \qquad (1.2)$$

Here m_k, μ_k, and λ_k are quantities which correspond to the k-th water-bearing and slightly permeable strata, x and r are the coordinates of the region, and q_i (x, t) and q_i(r, t) are the corresponding discharges in plane parallel ($\nu = 0$) and axisymmetrical ($\nu = 1$) cases of movement, which are expressed according to the Darcy law in dimensionless quantities as follows:

$$q_i = \xi^\nu \partial h_i / \partial \xi \qquad (\nu = 0,1) \qquad (1.3)$$

If in the plane parallel case the system (1.1) multiplied by $k_i m_i$ is differentiated with respect to x and in the case of axisymmetric flow system (1.3) is differentiated with respect to r and multiplied by $2\pi r k_i m_i$ and attention is paid to (1.3), for dimensionless discharges we will obtain a system of differential equations

$$\alpha_i \frac{\partial q_i}{\partial \tau} = a_i \Delta^* q_i - \beta_{i-1}(q_i - q_{i-1}) - \beta_i (q_i - q_{i+1}) + \varepsilon_i^\circ \quad (i = 1, 2, \ldots, n) \qquad (1.4)$$

where

$$\Delta^* = \frac{\partial^2}{\partial \xi^2} - \frac{\nu}{\xi}\frac{\partial}{\partial \xi}, \quad \varepsilon_i^\circ = a_i \xi^\nu \frac{\partial \varepsilon_i}{\partial \xi}, \quad q_0 = q_{n+1} = i_0(1 - \nu) \quad (\nu = 0, 1)$$

In deriving the system (1.4) in the axisymmetric case the gradient i_0 is assumed to be equal to 0.

Subsequently, in solving problems with given values of the discharge on the boundaries, the regions of flow will proceed from system (1.4).

We will now assume that at the initial moment of time $\tau = 0$ the functions of the discharge q_i are equal to certain arbitrary constant magnitudes q_{i0}. By using the Laplace transform with respect to time we change over from the initial to the representative region

$$Q_i = \int_0^\infty q_i e^{-p\tau} d\tau$$

and we obtain the following system of inhomogeneous ordinary differential equations:

$$a_i \Delta^* Q_i - (\alpha_i p + \beta_{i-1} + \beta_i) Q_i + \beta_{i-1} Q_{i-1} + \beta_i Q_{i+1} = -\alpha_i q_{i0} - F_i(\xi, p) \qquad (1.5)$$

where

$$F_i(\xi, p) = a_i \xi^\nu \int_0^\infty \frac{\partial \varepsilon_i}{\partial \xi} e^{-p\tau} d\tau \quad (\nu = 0, 1)$$

A general solution of system (1.5) consists of the sum of a particular and a general solution of the corresponding homogeneous system.

If we restrict ourselves to examination of movement in infinite strata with limited discharges at infinity, then the following functions are the solution to the corresponding homogeneous system:

$$Q_i = \begin{cases} A_i \exp(-\omega\xi) & \text{(in the plane parallel case)} \\ A_i \xi K_1(\omega\xi) & \text{(in the axisymmetric case)} \end{cases} \tag{1.6}$$

where $K_1(\omega\xi)$ is the Bessel function of the first order of the second kind with an imaginary argument.

Having substituted the values Q_i from (1.6) into the corresponding homogeneous system, we obtain the following system of algebraic equations for A_i:

$$(a_i\omega^2 - \alpha_i p - \beta_{i-1} - \beta_i)A_i + \beta_{i-1}A_{i-1} + \beta_i A_{i+1} = 0$$
$$(A_0 = A_{n+1} \equiv 0) \tag{1.7}$$

For the existence of a nontrivial solution of A_i of this system, its determinant must be equal to zero. Hence we will obtain for ω^2 a characteristic equation in the form of the determinant

$$|\Lambda_{kl}| = 0 \qquad (k, l = 1, 2, \ldots, n) \tag{1.8}$$

The elements of this determinant which are not equal to zero are situated along the main diagonal and also along the two symmetrical diagonals:

$$\{\Lambda_{12}, \Lambda_{23}, \Lambda_{34}, \cdots, \Lambda_{n-1,n}\} \text{ and } \{\Lambda_{21}, \Lambda_{32}, \Lambda_{43}, \cdots, \Lambda_{n,n-1}\}$$

and they are determined by the following formulas:

$$\Lambda_{kk} = a_k^2\omega^2 - \alpha_k p - \beta_{k-1} - \beta_k \quad (k = 1, 2, \ldots, n),$$
$$\Lambda_{k,k+1} = \Lambda_{k+1,k} = \beta_k \quad (k = 1, \ldots, n-1)$$

V. N. Émikh [4] has shown that when $p=0$ (a case of steady movement of underground waters), Eq. (1.8) has n simple positive roots. Similarly it can be shown that when $0 \le \operatorname{Re} p \le \infty$, Eq. (1.8) has also n simple positive roots ω_k^2, $k=1, 2, \ldots, n$.

Hence for each root ω_k^2 of Eq. (1.8) the rank of the matrix of the system (1.7) is equal to $n-1$, and for each ω_k^2 the system (1.7) has the solution $\{A_{1k}, \ldots, A_{nk}\}$, determined up to an accuracy of the constant A_{1k}. Then $A_{ik} = B_{ik}A_{1k}$.

A particular solution of $Q_i^\circ(\xi, p)$ of the inhomogeneous system is determined by a method of variation of constants.

Hence the solution of the inhomogeneous system (1.5) is written as

$$Q_i = \sum_{k=1}^{n} A_{1k}B_{ik}\xi^\nu\Theta_\nu + Q_i^\circ(\xi, p)$$
$$\Theta_0 = e^{-\omega_k\xi}, \quad \Theta_1 = K_1(\omega_k\xi) \quad (i = 1, \ldots, n) \quad (\nu = 0, 1) \tag{1.9}$$

In order to determine the arbitrary A_{1k}, it is necessary to assign n conditions. For example, in exploiting the strata of a well, it is necessary to assign the discharges of the well in each stratum.

The solution of system (1.4) is determined by the changeover operation from the representative function Q_i to the functions of the initial q_i. Hence integration of the expression (1.3) from ξ_0 to ξ gives correspondingly

$$h_i(\xi, \tau) = \int_{\xi_0}^{\xi} \frac{q_i(\xi, \tau)}{\xi^\nu} d\xi + h_i(\xi_0, \tau) \quad (\nu = 0, 1) \tag{1.10}$$

Meanwhile the arbitrary functions $h_i(\xi_0, \tau)$ can be treated as values of the pressure heads, where $\xi = \xi_0$. Let the functions $h_i(\xi, \tau)$ from (1.10) satisfy the system (1.1) in cases of plane parallel and axisymmetric movement. Then substituting $h_i(\xi, \tau)$ from (1.10) into the system (1.1), where $\xi = \xi_0$, in order to determine the functions $h_i(\xi_0, \tau)$ we arrive at the system of equations

$$\alpha_i \frac{dh_i}{d\tau} + \beta_{i-1}(h_i - h_{i-1}) + \beta_i(h_i - h_{i+1}) = \varepsilon_i(\xi_0, \tau) + f_i(\xi_0, \tau) \tag{1.11}$$

Here

$$f_i(\xi_0, \tau) = \frac{1}{\xi_0^\nu}\left(\frac{\partial q_i}{\partial \xi}\right)_{\xi=\xi_0}, \quad h_0 = h_{00} + i_0 x_0(1-\nu), \quad h_{n+1} = h_{n+1,0} + i_0 x_0 (1-\nu)$$
$$(\nu = 0, 1)$$

A particular solution of the system (1.11), corresponding to the initial condition $h_i(\xi_0, 0) = h_{i0}$, will be sought by an operational method.

Introducing a new function $y_i = h_i - h_{i0}$ and designating $h_{i-1,0} - h_{i,0}$ by $y_{i-1,i}$ for the representation

$$H_i(\xi_0, \tau) = \int_0^\infty y_i(\xi_0, \tau) e^{-p\tau} d\tau$$

we obtain a system of algebraic equations

$$(\alpha_i p + \beta_{i-1} + \beta_i) H_i - \beta_{i-1} H_{i-1} - \beta_i H_{i+1} = b_{i-1,i} p^{-1} + F_i(p) + E_i(p) \qquad (1.12)$$

where

$$F_i(p) = \int_0^\infty f_i(\xi_0, p) e^{-p\tau} d\tau, \; E_i(p) = \int_0^\infty \varepsilon_i(\xi_0, \tau) e^{-p\tau} d\tau, \; b_{i-1,i} = \beta_{i-1} y_{i-1,i} - \beta_i y_{i,i+1}$$

In determining the system (1.12), D_n is distinct from zero at any Re $p \geq 0$, since D_n has n simple negative roots p_k [4], and consequently the system (1.12) can be solved.

Hence, from (1.10) to (1.12) we obtain a solution of the system (1.1).

We note that when applying the Laplace transform with respect to τ to the functions $h_i(\xi, \tau)$ and when changing the order of integration on the right-hand side of the expression (1.10), we obtain on the basis of the representation (1.9)

$$H_i(\xi, p) = \sum_{k=1}^{n} \frac{B_{ik} A_{1k}}{\omega_k} \Omega_\nu(\xi) + \int_{\xi_0}^{\xi} \frac{Q_i^\circ(\xi, p)}{\xi^\nu} d\xi + H_i(\xi_0, p) \quad (\nu = 0, 1)$$

$$\Omega_0(\xi) = e^{-\omega_k \xi_0} - e^{-\omega_k \xi}, \qquad \Omega_1(\xi) = K_0(\omega_k \xi_0) - K_0(\omega_k \xi)$$

Here $K_0(\omega \xi)$ is the Bessel function of the zero order of the second kind. In the same way, the determination of the base functions $h_i(\xi, \tau)$ in the given case is reduced to an operation of changeover from the representations $H_i(\xi, \tau)$ to the initial $h_i(\xi, \tau)$.

2. In the system of nonpressure-head/pressure-head interacting water-bearing strata with limited capacities, the magnitude $\alpha_i = \sigma_i / \sigma_1$ for pressure strata (i = 2, 3, ..., n) is a small magnitude of a high order, which in a first approximation can be assumed to be $\alpha_i = 0$ for i = 2, 3, ..., n ($\alpha_1 = 1$).

It is not difficult to show that with an allowance of this kind, the determinant of the system (1.12) is distinct from zero at any Re $p \geq 0$.

In the case examined, the solution of the system (1.11) is determined directly by the sequence of the following functions:

$$h_1(\xi_0, \tau) = H_1 - e^{-b\tau} \left[H_1 - h_{10} - \int_0^\tau \left(\varepsilon(\xi_0, \tau) + \frac{1}{\varphi_1(n)} R_1(\xi_0, \tau) \right) e^{b\tau} d\tau \right] \qquad (2.1)$$

$$h_i(\xi_0, \tau) = \frac{\varphi_i(n)}{\varphi_{i-1}(n)} h_{i-1} + \frac{1}{\beta_{i-1} \varphi_{i-1}(n)} (R_i + h_{n+1}) \qquad (i = 2, \ldots, n) \qquad (2.2)$$

Here

$$b = \beta_0 + \frac{1}{\varphi_1(n)}, \quad \varphi_i(n) = \sum_{k=i}^{n} \frac{1}{\beta_k}, \quad H_1 = \frac{1}{b} \left[\beta_0 h_0 + \frac{h_{n+1}}{\varphi_1(n)} \right]$$

$$R_i(\xi_0, \tau) = \sum_{k=i}^{n} f_k(\xi_0, \tau) \varphi_k(n) \qquad (\varphi_n(n) = 1) \qquad (2.3)$$

We note that the function $h_1(\xi_0, \tau)$ is a solution of the equation

$$\frac{dh_1}{d\tau} + b(h_1 - H_1) = \frac{1}{\varphi_1(n)} R_1(\xi_0, \tau) + \varepsilon(\xi_0, \tau) \qquad (2.4)$$

in the case of the initial condition $h_1 = h_{10}$.

The given equation (2.4) together with the recurrent relationships (2.2) comprises a system identical to the system (1.11) in the case where $\alpha_1 = 1$, $\alpha_i = 0$, i = 2, ..., n.

From the solution of (2.1) and (2.2) it is possible to make some derivations, relating both to the qualitative and to the quantitative analysis of the question of interaction of water-bearing strata in the case of irregular movements.

1. Since the function

$$f_k(\xi_0, \tau) = \frac{1}{\xi_0^{\nu}} \left(\frac{\partial q_k}{\partial \xi}\right)_{\xi \to \xi_0}$$

is the speed of variation of discharges according to the coordinate in the fixed point ($\xi_0 \neq 0$) of space at any moment of time τ, then it is evident that the variation f_k (ξ_0, τ) turns out to be greatest in that stratum (or those strata) from which the evacuation takes place. For example, if the reduction in level takes place only in the first pressureless stratum, then among the f_k (ξ_0, τ) functions the greatest will be f_1 (ξ_0, τ).

On the other hand, it is seen from the expressions (2.1) to (2.3) that it consists of the sum of the integrals of the infiltration functions ε (ξ_0, τ) and the functions of the rate of variation of the discharges f_k (ξ_0, τ), in which before each function f_k (ξ_0, τ) stands the multiplier

$$\frac{\varphi_k(n)}{\varphi_{i-1}(n)} = \frac{\beta_k^{-1}+\ldots+\beta_n^{-1}}{\beta_{i-1}^{-1}+\ldots+\beta_n^{-1}} \qquad (k \geqslant i) \tag{2.5}$$

which decreases proportionally with the growth of k. In the given case, for h_1 (ξ_0, τ) the unit multiplier stands before f_1 (ξ_0, τ), but the multiplier $1/\varphi_1$ (n) stands before f_n (ξ_0, τ). Consequently, the function of the pressure head h_1 (ξ_0, τ) consists of components which reflect the variation of the function of discharge of each stratum in an interval of time [0, τ), whereby the influence of more distant water-bearing levels on the dynamics of the given stratum decreases in proportion to the number of slightly permeable strata according to Eq. (2.5).

It is possible to arrive at similar conclusions also for the other functions h_i (ξ_0, τ). Thus the disturbance taking place in the first stratum without a pressure head, decreasing $1/\varphi_1$ (n) times, is transmitted into the n-th pressure level. Such an analysis is most graphically obtained if for all β_i the condition $\beta_i = 1$ is fulfilled, which indicates that the weakly permeated strata have the same relationship λ_i/μ_i. Then

$$\frac{\varphi_k(n)}{\varphi_{i-1}(n)} = \frac{n-k+1}{n-i+2}, \quad b = \beta_0 + \frac{1}{n}$$

Hence the form of the functions h_i (ξ_0, τ) is somewhat simplified. If it is considered that the permeability of the slightly permeable strata decreases with increase in their depth under the surface of the earth, then, as can be seen from (2.5), this magnitude decreases even more with increase of k, and in the case of slight lowering or raising of the level in the pressureless stratum (as a result of using horizontal drainage or irrigation) the variations in the pressure levels in deeper strata will be negligible.

2. If in the absence of evacuation through a free surface of the pressureless stratum, infiltration feed of the same region enters the system of strata ($\varepsilon > 0$ corresponds to infiltration, $\varepsilon < 0$ corresponds to evaporation, $\beta_0 = 0$), then the whole of f_k (ξ_0, τ) = 0, i.e., internal disturbance is absent in the strata, and the function of the pressure heads will assume the form

$$\begin{aligned} h_1(\xi_0, \tau) &= h_{n+1} - e^{-b\tau}\left[h_{n+1} - h_{10} - \int_0^{\tau} \varepsilon(\xi_0, \tau)\, e^{b\tau} d\tau\right] \\ h_i(\xi_0, \tau) &= \frac{\varphi_i(n)}{\varphi_{i-1}(n)} h_{i-1}(\xi_0, \tau) + \frac{h_{n+1}}{\beta_{i-1}\varphi_{i-1}(n)} \end{aligned} \tag{2.6}$$

It is easily seen from (2.6) that the amplitude of the variation given by the infiltration function ε (ξ_0, τ), with removal downwards from the surface of the earth, decreases in proportion to the number of slightly permeable strata with the number of parameters corresponding to it.

We note that the derivations made here relate only to the case of an assumption that the pressure head h_{n+1} is constant in (n + 1)-th pressure level.

It is evident that small local variations taking place in adjoining, especially distant strata do not show a considerable quantitative variation in the dynamics of the water-bearing sandy levels when they have a determined gradient, a remote lateral feed source, and a sufficiently high capacity. The stating and solution of problems of underground water are possible assuming that the pressure head is constant in similar strata, when evacuation from them does not occur, since an allowance of this kind leads only to simplification of the mathematical model of the phenomenon without changing its nature.

3. Cases occur in which the system of strata, located in the upper parts of the earth's crust, is limited from below by impervious rocks and forms a single reservoir of interconnected water-bearing levels.

The solution of the problem corresponding to this case is obtained from systems (2.1) to (2.3), where $\beta_n \to 0$:

$$h_1(\xi_0,\tau) = h_0 - e^{-\beta_0\tau}\left[h_0 - h_{10} - \int_0^\tau \left(\sum_{k=0}^{n} f_k(\xi_0,\tau) + \varepsilon(\xi_0,\tau)\right) e^{\beta_0\tau} d\tau\right]$$
$$h_i(\xi_0,\tau) = h_{i-1}(\xi_0,\tau) + \frac{1}{\beta_{i-1}} \sum_{k=1}^{n} f_k(\xi_0,\tau) \tag{2.7}$$

It is seen from (2.7) that, as distinct from the previous case, the absence of a constant feed source from below modifies the result here somewhat in the quantitative sense. Now $h_1(\xi_0, \tau)$ consists of components which reflect the variation of the functions of the discharges of each stratum of the system in an interval of time $[0, \tau)$ in which the influence of each of them on the level without a pressure head is transmitted without any changes from the side of the slightly porous strata.

We will indicate one more formula which is obtained from system (2.7) by means of successive addition:

$$h_n(\xi_0,\tau) = h_1(\xi_0,\tau) + \sum_{k=2}^{n} f_k(\xi_0,\tau) \sum_{l=i}^{k-1} \frac{1}{\beta_l} \tag{2.8}$$

Any variation taking place in the pressureless stratum at the moment of time τ is directly transmitted to the lower level into the reservoir of interconnected strata with impervious water support by taking into account the rigidity of the system of pressure levels, as shown by (2.8). However, here, as in the previous case, the influence of any local internal process (evacuation, etc.) is transmitted to the other strata of the reservoir fully dependent on the degree of water permeability of slightly permeable clay interstratifications.

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